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ON NILPOTENT PROPERTIES OF LEIBNIZ ALGEBRAS

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A generalization of a classical result from the theory of nilpotent Lie algebras to Leibniz algebras leads to several applications concerning the nilpotent properties both of these two types of algebras.

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1. INTRODUCTION

Leibniz algebras are introduced by Loday (1993) as nonantisymmetric generalization of Lie algebras. Some structure results concerning nilpotency of Leibniz algebras are due to Ayupov and Omirov (1998). The main result in this article (Theorem 7) can be considered as a generalization of a well-known result that if a finite-dimensional Lie algebra acts on a vector space by nilpotent endomorphisms, then these have at least one common eigenvector corresponding to their unique eigenvalue zero.

Definition 1. An algebra A over a field F is called a *Leibniz algebra* if its multiplication satisfies the Leibniz identity

$$(ab)c = a(bc) - b(ac)$$

for all $a, b, c \in A$.

This is in fact a left Leibniz algebra. A right Leibniz algebra must satisfy the dual identity

$$a(bc) = (ab)c - (ac)b$$

for all $a, b, c \in A$.

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In this article we are considering only left Leibniz algebras. We remark that if a left Leibniz algebra A satisfies the additional identity $aa = 0, a \in A$, then the Leibniz identity is equivalent to the Jacobi identity

$$a(bc) + b(ca) + c(ab) = 0.$$

Hence, Lie algebras are particular cases of Leibniz algebras.

If B is a subspace of a Leibniz algebra A, then we define the subspaces B^n and B_n for any integer $n \geq 1$ by the formulas

$$B^1 = B, \ldots, B^n = BB^{n-1}$$

and

$$B_1 = B, \ldots, B_n = \sum_{p=1}^{n-1} B_p B_{n-p}.$$

We have the following lemma.

Lemma 2. *If A is a Leibniz algebra and if B is a subspace of A, then*:

(i) $B^m B^n \subseteq B^{m+n}$, $m, n \geq 1$;
(ii) $B_n = B^n$, $n \geq 1$.

Proof (Ayupov and Omirov, 1998). In the case $B = A$, each subspace A^n is both a left and a right ideal of A. Indeed, A^n is obviously a left ideal and, from the inclusion (i) of Lemma 2, we deduce that $A^n A \subseteq A^{n+1} \subseteq A^n$, $n \geq 1$. □

Definition 3. A Leibniz algebra A is nilpotent if there exists an integer k such that any product $a_1 a_2 \ldots a_k$ of k elements of A, no matter how associated, is equal to zero.

This definition expressed in terms of the series of $(A_n)_{n\geq 1}$ states that A is nilpotent if there exists an integer k such that $A_k = 0$. From equation (ii) of Lemma 2 we deduce that a Leibniz algebra A is nilpotent if there exists an integer k such that the ideal $A^k = 0$.

Definition 4. A bimodule of a Leibniz algebra A (cf. Loday and Pirashvili, 1996) is a vector space M over F equipped with two bilinear compositions denoted by ma and am, for any $a \in A$ and $m \in M$, such that

$$(ma)b = m(ab) - a(mb)$$

$$(am)b = a(mb) - b(am)$$

$$(ab)m = a(bm) - b(am)$$

We denote by $End(M)$ the associative algebra of all endomorphisms of the vector space M. If M is a bimodule of A, then each of the mappings $S_a : m \to ma$ and

$T_a : m \to am$ is an endomorphism of M, and the mappings $S_a : a \to S_a$, $T_a : a \to T_a$ of A into $End(M)$ are linear.

Definition 5. A representation of a Leibniz algebra A on a vector space M is a pair (S, T) of linear mappings $a \to S_a$, $T_a : a \to T_a$ of A into $End(M)$ such that

$$\begin{aligned} S_b \circ S_a &= S_{ab} - T_a \circ S_b \\ S_b \circ T_a &= T_a \circ S_b - S_{ab} \\ T_{ab} &= T_a \circ T_b - T_b \circ T_a \end{aligned} \tag{1}$$

for all $a, b \in A$.

It follows that the vector space M equipped with the compositions $ma = S_a(m)$ and $am = T_a(m)$ is a bimodule of A. Clearly, the concepts representation and bimodule are equivalent.

Let A be a Leibniz algebra. The right multiplication R_a (resp., the left multiplication L_a) of A determined by any element $a \in A$ is the endomorphism of A defined by $R_a(x) = xa$ (resp., $L_a(x) = ax$) for all $x \in A$. The pair (R, L) of the linear mappings $R : a \to R_a$ and $L : a \to L_a$ is a representation of A on A itself.

Now, we recall some basic notions of the theory of Lie algebras. Let M be a vector space over F. If $X, Y \in End(M)$, then their Lie bracket is defined as $[X, Y] = X \circ Y - Y \circ X$. The vector space $End(M)$ equipped with this bracket becomes a Lie algebra denoted by $gl(M)$.

A representation of a Lie algebra $\mathfrak{g}$ on the vector space M is simply a map of Lie algebras $\rho : \mathfrak{g} \to gl(M)$. The representation $ad : gl(M) \to gl\,(gl(M))$ of the Lie algebra $gl(M)$ on itself defined by $ad(X)(Y) = [X, Y]$ is called the *adjoint representation*.

To return to our Definition 5, by the third equation of (1), the subspace $T(A)$ of $gl(M)$ is closed under the Lie bracket. Hence, $T(A)$ is a subalgebra of the Lie algebra $gl(M)$.

Another consequence of the Definition 5 is the fact that the subspace $S(A) \subset gl(M)$ is a $T(A)$-module via the adjoint representation. Indeed, the adjoint action $ad(T(A))$ of $T(A)$ on the vector space $gl(M)$ preserves its subspace $S(A)$, since the second equation of (1) implies that

$$ad(T_a)(S_b) = [T_a, S_b] = S_{ab}, \qquad a, b \in A.$$

Finally, we note that any representation $\rho : \mathfrak{g} \to gl(M)$ of a Lie algebra, extended by the zero mapping $0 : \mathfrak{g} \to gl(M)$, can be always considered as a representation of $\mathfrak{g}$ viewed as a Leibniz algebra.

2. THE MAIN RESULT AND ITS CONSEQUENCES

The first two equations of (1) imply the following:

$$S_b \circ S_a + S_b \circ T_a = 0, \qquad a, b \in A.$$

For $a = b$, we obtain the equation

$$S_a^2 = -S_a \circ T_a, \qquad a \in A. \tag{2}$$

This is a particular case of the following lemma.

Lemma 6. *For $n \geq 1$, we have*

$$S_a^n = (-1)^{n-1} S_a \circ T_a^{n-1}, \qquad a \in A$$

Proof. The proof follows by induction on $n \geq 1$, the cases $n = 1$ and $n = 2$ being clear. We suppose that the formula of the lemma is true for $n = k$, and we have

$$S_a^{k+1} = (-1)^{k-1} S_a^2 \circ T_a^{k-1}.$$

Thus, by (2),

$$S_a^{k+1} = (-1)^k S_a^2 \circ T_a^k,$$

and the proof of Lemma 6 is complete.

Our main result is the following theorem.

Theorem 7. *Let A be a finite-dimensional Leibniz algebra, and let (S, T) be a representation of A on a vector space M such that T_a is a nilpotent endomorphism of M for any $a \in A$. Then*:

(i) *S_a is a nilpotent endomorphism of M for any a in A*;
(ii) *There exists a nonzero vector m in M such that for all $a, b \in A$*

$$T_a(m) = S_b(m) = 0.$$

Proof. (i) Since T_a is a nilpotent endomorphism of M for any $a \in A$, the proof follows immediately from the Lemma 6.

(ii) The proof follows by induction on $\dim A$. The theorem is obvious if $\dim A = 0$. We suppose that this is true for all Leibniz algebras of dimension $< n$. Now, let A be a Leibniz algebra with $\dim A = n$, and let (S, T) be a representation of A on a vector space M ($M \neq 0$) such that T_a is a nilpotent endomorphism of M for any $a \in A$. We have $\dim T(A) \leq n$.

First we consider the case $\dim T(A) < n$. Since $T(A)$ is a subalgebra of the Lie algebra $gl(M)$ consisting of nilpotent endomorphisms of M, the induction hypothesis implies that the vectors $m \in M$ such that $T_a(m) = 0$, for any $a \in A$, form a nonzero subspace W of M. Clearly, it suffices to show that there is a nonzero vector in M killed by S_b for all $b \in A$.

We observe that this is true if there exists an element S_d in $S(A)$ carrying W to itself and its restriction on W is nonzero. Indeed, there exists a nonzero vector m_0 in W such that $S_d(m_0)$ is also a nonzero vector of W killed by S_b for all $b \in A$, because $S_b \circ (S_d + T_d) = 0$.

Now we give the following algorithm.

For notational uniformity, let us put $N_0 = 0$.

For $i = 1, \dots, \dim S(A)$, we write out the general step.

Let N_{i-1} be a subspace of $S(A)$ of dimension $i-1$ consisting of elements carrying W to 0, which is stable under $ad(T_a)$ for any a in A. By passing to the quotient, the Lie algebra $T(A)$ acts on $S(A)/N_{i-1}$. The induction hypothesis assures the existence of S_y in $S(A)$, $S_y \notin N_{i-1}$, such that $ad(T_a)(S_y) = S_{ay} \in N_i$ for any $a \in A$.

It follows that W is stable under S_y. Indeed, for any $a \in A$,

$$T_a\left(S_y(m)\right) = S_{ay}(m) + S_y\left(T_a(m)\right) = 0.$$

If the restriction of S_y on W is different from zero, then by our observation above, the proof is complete and the algorithm stops. If S_y carries W to 0, then we set $N_i = N_{i-1} \oplus F.S_y$.

The algorithm gives as output a subspace N of $S(A)$ consisting of operators carrying W to 0, and an operator S_z carrying W to itself, such that $S(A) = N \oplus F.S_z$.

It follows immediately that if S_z maps W to 0, then each element of $S(A)$ does the same. Thus, in both cases either S_z maps W to 0 or the restriction of S_z on W is different from zero the proof of the theorem with $\dim T(A) < n$ is completed.

In the remaining case $\dim T(A) = n$, since the linear mapping $T : A \to T(A)$ is an isomorphism, we may consider the algebra A as a Lie algebra isomorphic to $T(A)$.

First, we show that A contains an ideal of codimension one. Let $\mathfrak{h}$ be a subalgebra of the Lie algebra $T(A)$ with $\dim \mathfrak{h} < n$. The adjoint action $ad(\mathfrak{h})$ of $\mathfrak{h}$ on $gl(M)$ preserves the subspace $T(A) \subset gl(M)$. Thus, by passing to the quotient, $\mathfrak{h}$ acts on $T(A)/\mathfrak{h}$. Since any $X \in \mathfrak{h}$ is a nilpotent endomorphism of M, $ad(X)$ acts nilpotently on $gl(M)$, hence on $T(A)/\mathfrak{h}$. By the induction hypothesis, there exists $T_e \in T(A)$, $T_e \notin \mathfrak{h}$, such that

$$ad(T_e)(T_a) = [T_e, T_a] = T_{ea} \in \mathfrak{h}$$

for all $a \in A$. This implies that the subspace of $T(A)$ spanned by $\mathfrak{h}$ and T_e is a subalgebra containing $\mathfrak{h}$ as an ideal of codimension one. Hence, we conclude by induction (starting from $\mathfrak{h} = \{0\}$) that $T(A)$ contains an ideal $\mathfrak{M}$ of codimension one, that is $T(A) = \mathfrak{M} \oplus F.T_e$ for some $e \neq 0$ in A. It follows that the Lie algebra A contains the set

$$I = \{x \in A : T_x \in \mathfrak{M}\}$$

as an ideal of codimension one. Let us note that $A = I \oplus F.e$.

We now turn to the representation (S, T) of A in M. By applying the induction hypothesis to the ideal I of A, we conclude that there exists a nonzero vector m in M such that $T_x(m) = S_y(m) = 0$ for all $x, y \in I$; let $W \subseteq M$ be the nonzero subspace of all such vectors $m \in M$.

Next, we show that W is stable under T_e and S_e. Indeed, from equations (1) and the fact that I is an ideal of the Lie algebra A, we obtain

$$T_x(T_e(m)) = T_{xe}(m) - T_e(T_x(m)) = 0$$
$$S_y(T_e(m)) = S_{ye}(m) - T_e(S_y(m)) = 0$$
$$T_x(S_e(m)) = S_{xe}(m) - S_e(T_x(m)) = 0$$
$$S_y(S_e(m)) = S_{ey}(m) - T_e(S_y(m)) = 0$$

for all $x, y \in I$ and $m \in W$.

Since T_c is nilpotent, there exists $m_0 \neq 0$ in W such that $T_e(m_0) = 0$. We have two cases:

(i) If $S_e(m_0) = 0$, then clearly m_0 is killed by T_a and S_b for all $a, b \in A$;
(ii) If $S_e(m_0) \neq 0$ in W, then $S_e(m_0)$ itself is killed by T_a and S_b for all $a, b \in A$, because of the relations

$$T_e(S_e(m_0)) = S_{ee}(m_0) - S_e(T_e(m_0)) = 0$$
$$S_e(S_e(m)) = -S_e(T_e(m_0)) = 0,$$

and the fact that the relation $ee = 0$ implies that $S_{ee} = 0$.

In the particular case of Lie algebras, we deduce immediately from Theorem 7 the following well-known result.

Corollary 8. *Let $\mathfrak{g}$ be any subalgebra of the Lie algebra $gl(M)$, such that every X in $\mathfrak{g}$ is a nilpotent endomorphism of M. Then there exists a nonzero vector $m \in M$ such that $X(m) = 0$ for all $X \in \mathfrak{g}$.*

Another useful application of Theorem 7 is the following corollary.

Corollary 9. *Let A be a finite-dimensional Leibniz algebra. Let (S, T) be a representation of A on a finite-dimensional vector space M such that T_a is a nilpotent endomorphism of M for any $a \in A$. Then, there exists a basis of M relative to which the matrix representative of T_a and S_b is strictly upper triangular for all $a, b \in A$.*

Proof. In a standard way, if we begin with any nonzero vector m_1 in M such that $T_a(m_1) = S_b(m_1) = 0$ for all $a, b \in A$, then, since the induced action of A on the quotient of M by the span $\langle m_1 \rangle$ of m_1 is also by nilpotent endomorphisms, we can find a vector m_2 in M ($m_1 \notin \langle m_1 \rangle$) such that both $T_a(m_2)$ and $S_b(m_2)$ belong to $\langle m_1 \rangle$ for all $a, b \in A$. Next, we consider the span $\langle m_1, m_2 \rangle$ of m_1 and m_2 and so on. After $\dim M$ repetitions of the above procedure, we arrive to the desired basis of M.

Now, from Corollary 9, we deduce the following one.

Corollary 10. *A finite-dimensional Leibniz algebra A is nilpotent if and only if L_a is a nilpotent endomorphism of A for any $a \in A$.*

Proof. Let A be nilpotent. Then there exists an integer k such that $A^k = 0$. Since $L_a^{k-1}(b) = 0$ for all $a, b \in A$, L_a is a nilpotent endomorphism of A for any $a \in A$.

Conversely, by applying Corollary 9 to the representation (R, L), we find a basis of A with respect to which each endomorphism L_a is represented by strictly upper triangular matrices. It follows that if we set $n = \dim A$, then any composition of at least $n - 1$ elements of $L(A)$ is equal to zero. In other words $A^n = 0$, and therefore A is nilpotent.

In particular, if we apply Corollary 9 to a finite-dimensional Lie algebra and its adjoint representation, then we immediately obtain the well-known Engel's Theorem.

Corollary 11. *A finite-dimensional Lie algebra* $\mathfrak{g}$ *is nilpotent if and only if* adx *is nilpotent for any* x *in* $\mathfrak{g}$.

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